

An information theoretical analysis of quantum optimal control

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We show that if an efficient classical representation of the dynamics exists, optimal control problems on many-body quantum systems can be solved efficiently with finite precision. We show that the size of the space of parameters necessary to solve quantum optimal control problems defined on pure, mixed states and unitaries is polynomially bounded from the size of the set of reachable states in polynomial time. We provide a bound for the minimal time necessary to perform the optimal process given the bandwidth of the control pulse, that is the continuous version of the Solovay-Kitaev theorem. We explore the connection between entanglement present in the system and complexity of the control problem, showing that one-dimensional slightly entangled dynamics can be efficiently controlled. Finally, we quantify how noise affects the presented results.

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Quantum optimal control lies at the heart of the modern quantum revolution, as it allows to match the stringent requirements needed to develop quantum technologies, to develop novel quantum protocols and to improve their performances [1]. Along with the increased numerical and experimental capabilities developed in recent years, problems of increasing complexity have been explored and recently a lot of attention has been devoted to the application of optimal control (OC) to many-body quantum dynamics: OC has been applied to information processing in quantum wires [2], the crossing of quantum phase transitions [3], the generation of many-body squeezed or entangled states [4], chaotic dynamics [5], unitary transformations [6]. Recent studies have been devoted to the understanding of the fundamental limits of OC in terms of energy-time relations (time-optimal) [7] and its robustness against perturbations [8, 9].

These exciting developments call for the development of a general framework to understand when and under which conditions is it possible to solve a given OC problem in a many-body quantum system. Indeed, due to the exponential growth of the Hilbert space with the number of constituents, solving an OC problem on a many-body system is in general highly inefficient: the algorithmic complexity (AC) of exact time-optimal problems can be super-exponential [6]. However, limited precision, errors and practical limitations naturally introduce a finite precision both in the functional to be minimized and on the total time of the transformation. The smoothed complexity (SC) has been introduced recently to cope with this situation to describe the “practical” complexity of solving a problem in the real world with finite precision. It has been shown that the SC can be drastically different from the AC: indeed the AC –which is defined by the scaling of the worst case– might be practically irrelevant as the worst case might be never found in practice [10]. A paradigmatic case is that of the simplex algorithm applied to linear programming problems: it is characterized by an exponential AC in the dimension of the searched space, however the SC is only polynomial, that is, the

worst case disappears in presence of perturbations [11].

In this letter, we perform an information theoretical analysis providing a first step towards the theoretical understanding of the complexity of OC problems in many-body quantum systems. We present a counting argument to bound the size of the space of parameters needed to solve OC problems defined over the set of time-polynomially reachable states. We explore the implications of this result in terms of SC identifying some classes of problems that can be efficiently solved. We characterize the effects of noise in the control field and of the entanglement present during the system dynamics. We finally provide an information-time bound, relating the bandwidth of the control field with the minimal time necessary to achieve the optimal transformation.

A quantum OC problem can be stated as follows: given a dynamical equation

$$\dot{\rho} = \mathcal{L}(\rho, \gamma(t)); \quad (1)$$

with boundary condition $\rho(t=0) = \rho_0$ where ρ is the density matrix describing a quantum system defined on an Hilbert space $\mathcal{H} = \mathbb{C}^N$, and \mathcal{L} the Liouvillian operator with the unitary part generated by an Hamiltonian

$$H = H_D + \gamma(t)H_C, \quad (2)$$

where $\gamma(t)$ is a time-dependent control field, and H_D and H_C the drift and control Hamiltonian respectively. For simplicity here we consider the case where only a single control field is present (the generalization is straightforward) and we work in adimensional units. From now on we focus on finite-size Hilbert space of dimension N , as any quantum system with limited energy and limited in space is effectively finite-dimensional. Eq. (1) generates a set of states depending on the control field $\gamma(t)$ and on the initial state ρ_0 : the manifold that is generated for every $\gamma(t)$ defines the set of reachable states \mathcal{W} with dimension $D_{\mathcal{W}}(N)$ [12]. If the system is controllable –i.e. the operators H_D, H_C generate the complete dynamical Lie algebra– the manifold \mathcal{W} is the complete space of density

matrix operators and its dimension is $D_{\mathcal{W}} = N^2$ for an N -dimensional Hilbert space, where for n d-level quantum systems $N = d^n$. Given a goal state $\bar{\rho}$ the problem to be solved is to find a control pulse $\bar{\gamma}(t)$ that drives the system from a reference state ρ_0 within an ϵ -ball around the goal state $\bar{\rho}$. Equivalently, the OC problem can be expressed as a functional minimization of the form

$$\min_{\gamma(t)} \mathcal{F}(\rho_0, \bar{\rho}, \gamma(t), [\lambda_i]), \quad (3)$$

where the functional \mathcal{F} might also include constraints introduced via Lagrange multipliers λ_i . The functional \mathcal{F} is minimised by an (not necessary unique) optimal $\bar{\gamma}(t)$, that identifies a final state ρ_f such that $\|\rho_f - \bar{\rho}\| < \epsilon$ in some norm $\|\cdot\|$.

We now recall the definition of the information content of the control pulse $\gamma(t)$ as we show in the following that it is intimately related to the complexity of the OC problem. The information (number of bits b_γ) carried by the control pulse $\gamma(t)$ is given by the classical channel capacity C times the pulse duration T . In the simple case of a noiseless channel, the channel capacity is given by Hartley's law, thus

$$b_\gamma = T \Delta\Omega \kappa_s \quad (4)$$

where $\Delta\Omega$ is the bandwidth, and $\kappa_s = \log(1 + \Delta\gamma/\delta\gamma)$ is the bit depth of the control pulse $\gamma(t)$, and $\Delta\gamma = \gamma_{max} - \gamma_{min}$ and $\delta\gamma$ are the maximal and minimal allowed variation of the field [19]. Note that given an uniform sampling rate of the signal δt , $T \Delta\Omega = T/\delta t = n_s$ where n_s is the number of sampling points of the signal. Any optimization method of choice depends on these n_s variables, i.e. n_s defines the dimension of the input of the optimisation problem. We thus define the dimension of the quantum OC problem \mathcal{D} as follows: Given a dynamical law of the form of Eq.(1), a reference initial state ρ_0 and any possible goal state in the set reachable states \mathcal{W} , the dimension of the quantum OC problem is defined by the minimal number of independent degrees of freedom \mathcal{D} in the OC field necessary to achieve the desired transformation up to precision ϵ . Notice that \mathcal{D} might be the minimal number of sampling points n_s , of independent bang-bang controls, of frequencies present in the control field or the dimension of the subspace of functions the control field has non-zero projection on.

From now on we consider the physical situations where the control is performed in some finite time $t \in [0, T]$, with bounded control field and bounded Hamiltonians, e.g. $\|H_D\| = \|H_C\| = 1$ and $\gamma(t) \in [\gamma_{min} : \gamma_{max}] \forall t$. The aforementioned physical constraints, naturally introduce a new class of states, that we define as follows: The set of time-polynomial reachable states $\mathcal{W}^+ \subseteq \mathcal{W}$ is the set of states that can be reached (with finite energy) with precision ϵ in polynomial time as a function of the set size $D_{\mathcal{W}^+}(N) \leq D_{\mathcal{W}}(N)$. This is the class of interesting states from the point of view of OC, as if a state can be reached only in exponential time there is no need of OC at all: in exponential time any reachable state

is reached also with a constant Hamiltonian. Similarly to standard definitions, we define a time-polynomial reachable system if all states can be reached (with precision ϵ) in polynomial time by means of at least one path (i.e. $D_{\mathcal{W}^+} = D_{\mathcal{W}}$) and a time-polynomially controllable system if \mathcal{W}^+ is equal to the whole Hilbert space. Notice that if the bound on the strength of the control γ_{max} is relaxed we have $D_{\mathcal{W}^+} = D_{\mathcal{W}}$. Given the above definitions, we can state the following:

Theorem The size \mathcal{D} of a quantum OC problem in \mathcal{W}^+ up to precision ϵ is a polynomial function of the size of the manifold of the time-polynomial reachable states $D_{\mathcal{W}^+}$.

Proof We first prove that the dimension of the problem is bounded from below by $D_{\mathcal{W}^+}$ and then that is bounded from above by a polynomial function of $D_{\mathcal{W}^+}$.

Lower bound: We divide the complete set of time-polynomial reachable states \mathcal{W}^+ in balls of size $\epsilon^{D_{\mathcal{W}^+}}$. The number of ϵ -balls necessary to cover the whole set \mathcal{W}^+ is $\epsilon^{-D_{\mathcal{W}^+}}$ and one of them identifies the set of states that live around the state $\bar{\rho}$ within a radius ϵ . The information content of the OC field must be at least sufficient to specify the ϵ -ball surrounding the goal state, that is $b_\gamma \geq b_S^-$, where $b_S^- = \log \epsilon^{-D_{\mathcal{W}^+}}$. Finally one obtains

$$\epsilon \geq 2^{-\frac{T \Delta\Omega \kappa_s}{D_{\mathcal{W}^+}}}. \quad (5)$$

Setting a maximal precision (e.g. machine precision) expressed in bits $\kappa_\epsilon = -\log_2 \epsilon$ results in $n_s \kappa_s / D_{\mathcal{W}^+} = \kappa_\epsilon$; and imposing $\kappa_\epsilon = \kappa_s$ we obtain

$$n_s \geq D_{\mathcal{W}^+}. \quad (6)$$

Upper bound: The goal state belongs to the set of time-polynomial states $\bar{\rho} \in \mathcal{W}^+$, thus a path of finite length L that connects the initial and goal states in polynomial time exists. The maximum of (non-redundant) information that provides the solution to the problem is the information needed to describe the complete path b_S^+ . Setting the desired precision ϵ , this is equal to $\log \epsilon^{-D_{\mathcal{W}^+}}$ bit of information for each ϵ -ball needed to cover the path times the number of balls n_ϵ . The latter is given by

$$n_\epsilon = L/\epsilon \leq T v_{max}/\epsilon = Poly(D_{\mathcal{W}^+}) v_{max}/\epsilon \quad (7)$$

where L is the length of the path, v_{max} is the maximal allowed velocity along the path due to the bounded energy. In conclusion, we obtain that

$$b_S^+ = \frac{Poly(D_{\mathcal{W}^+}) v_{max}}{\epsilon} \log \epsilon^{-D_{\mathcal{W}^+}}, \quad (8)$$

that implies together with the condition $b_\gamma \leq b_S^+$

$$Poly'(D_{\mathcal{W}^+}) v_{max}/\epsilon \geq n_s \quad (9)$$

As n_s is bounded by a polynomial function of $D_{\mathcal{W}^+}$, thus $\mathcal{D} = Poly(D_{\mathcal{W}^+})$ ■

Notice that the lower bound holds in general for any reachable state in \mathcal{W} and can be saturated, as recently shown in [13]. On the other hand, the upper bound diverges for $\varepsilon \rightarrow 0$, as finding the exact solution of the control problem might be as difficult as super exponential [6]. The theorem has a number of interesting practical and theoretical implications that we present in the rest of the paper.

Complexity - The aforementioned theorem poses the basis to set the SC of solving the OC problem. An algorithm recently introduced to solve complex quantum OC problems, the Chopped RAndom Basis (CRAB) optimisation, builds on the fact that the space of the control pulse $\bar{\gamma}(t)$ is limited from the very beginning to some (small) value \mathcal{D} , and then solves the problem by means of a direct search method as the simplex algorithm. Recently, numerical evidence has been presented that this algorithm efficiently finds exponentially precise solutions as soon as $\mathcal{D} \geq D_{\mathcal{W}}$ [14]. This result can be put now on solid ground as under fairly general conditions OC problems are equivalent to linear programming [15] and linear programming can be solved via simplex algorithm with polynomial SC [10]: thus, the CRAB optimisation solves with polynomial SC OC problems with dimension \mathcal{D} . More formally, one can make the following statement: The class of OC problems that satisfy the hypothesis (H1-H3) of Ref. [15], is characterised by a polynomial SC in the size of the problem \mathcal{D} . In conclusion, studying the scaling of the dimension of the control problem $\mathcal{D} = \text{Poly}(D_{\mathcal{W}^+})$ is of fundamental interest to understand and classify our capability of efficiently control quantum systems. The first results in this direction can be obtained observing the influence of the integrability of the quantum system on $D_{\mathcal{W}^+}$, resulting in the following properties:

1 - The size \mathcal{D} of a generic OC problem defined on time-polynomial controllable non-integrable n -body quantum system is exponential with the number of constituents n . Indeed the dynamics of a controllable non integrable many-body quantum system explores the whole Hilbert space, i.e. the set of time-polynomial reachable states is the whole Hilbert space, that is $D_{\mathcal{W}^+} = N^2$ ($D_{\mathcal{W}^+} = N$ for pure states).

On the contrary, despite the exponential growth of the Hilbert space, the size of \mathcal{W}^+ for integrable systems is at most linear in the number n of constituents of the system, that implies together with the theorem above that:

2 - The size \mathcal{D} of OC problems defined on time-polynomially controllable integrable many-body quantum system, is polynomial with $n = \log_d(N)$. Notice that this statement generalizes a theorem that has been proven for the particular case of tridiagonal Hamiltonian systems presented in [16].

Finally, there exists a class of intermediate dynamics that despite in principle might explore an exponentially big Hilbert space, are confined in a corner of it and can thus be efficiently represented. The simplest example of this class of problems is mean-field dynamics, however

more generally, to this class of dynamics belongs for example those that can be represented efficiently by means of a tensor-network as t-DMRG [17]. We can thus state the following:

3 - The dimension \mathcal{D} of an OC problem defined on a dynamical process that can be described efficiently by a tensor network, e.g. in one dimension a matrix product state, is polynomial in the number of system components n . The dimension of the set of the time-polynomial reachable states \mathcal{W}^+ that can be efficiently represented by a tensor network scales as $D_{\mathcal{W}^+} \leq D_{\mathcal{W}} \leq \text{Poly}(n) \cdot T$ where T is the total time of the evolution and $\text{Poly}(n)$ is the dimension of the biggest tensor network state represented during the time evolution. Notice that, although the previous statement is in principle valid in all dimensions, it has practical implications mostly in one-dimensional systems as much less efficient representations of the dynamics are known in dimensions bigger than one [18].

We can now link directly the entanglement present in the system during its dynamics with the complexity of controlling it:

4 - Time evolution of slightly entangled one-dimensional many-body quantum systems can be efficiently represented via Matrix Product States with $D_{\mathcal{W}^+} \leq D_{\mathcal{W}} = O(T d 2^{2S} n)$ parameters, where S is the maximal Von Neumann entropy of any bipartition present in the system. Thus, systems with $S \propto \log(n)$ for every time can be efficiently controlled.

We stress that the entanglement present in the system is not uniquely correlated with the complexity of the OC problem: indeed due to the previous results, integrable systems (also highly entangled) are efficiently controllable, as shown recently in [13]. On the contrary, as said before, highly entangled dynamics of non integrable systems, for which it does not exist an efficient representation as $S \propto n$ are exponentially difficult to control. In conclusion, the size of the control problem depends on the dimension of the manifold over which the dynamics takes place. This can be simply understood by considering the scenario where the dynamics over which the control problem is defined is restricted to the space of two eigenstates of a complex many-body hamiltonian, each of them highly entangled w.r.t some local bases. If one has access to a direct coupling between them, the complexity of the OC problem is not more than that of a simple Landau-Zener process (independently from the entanglement present in the system) as the manifold is effectively two-dimensional. However, this is not generally the case, as one has usually access to some local (or global) operator, and the dynamic of the system is not in general restricted to two states. In the case of non integrable systems, a generic couple of initial and goal states projects on exponentially many basis states independently of the chosen basis, while for integrable states it exists a base where the states have a simple representation. Thus, the minimal amount of information needed to solve the quantum OC problem is exponential and polynomial respectively. In between, there is the class

of TN-efficiently representable dynamics, for which we know how to build an efficient representation and correspondingly we know how to efficiently solve the OC problem.

Time bounds - Manipulating Eq. (5) applied to the whole set of reachable states \mathcal{W} we achieve a bound for the minimal time needed to achieve the desired transformation as a function of the control bandwidth: The minimal time needed to reach a given final state in $D_{\mathcal{W}}$ with precision ε at finite bandwidth is

$$T \geq \frac{D_{\mathcal{W}}}{\Delta\Omega \kappa_S} \log(1/\varepsilon) \quad (10)$$

or again, under the assumption that $\kappa_\varepsilon = \kappa_s$:

$$T \geq \frac{D_{\mathcal{W}}}{\Delta\Omega}. \quad (11)$$

The previous relation is a continuous version of the Solovay-Kitaev theorem: it provides an estimate of the minimal time needed to perform an optimal process given a finite band-width. Notice also that the bandwidth provides the average bits rate per second, thus this results coincides with the intuitive expectation that the minimal time needed to perform an optimal quantum process is the time necessary to “inform” the system about the goal state given that the control field has only a finite bit transmission rate.

We recall that there is a time-energy bound, known as quantum speed limit that in its general form is

$$T_{QSL} \geq \frac{d(\rho_0, \rho_G)}{\bar{\Lambda}}, \quad (12)$$

where $d(\cdot, \cdot)$ is the distance and $\bar{\Lambda} = \int_0^T \|\mathcal{L}\|_p dt / T$ with $\|\cdot\|_p$ the p-norm [7]. The best efficient process saturates both bounds, that implies $\Delta\Omega \propto D_{\mathcal{W}}$; thus the bandwidth of the time-optimal pulse in general should scale as the dimension of the space \mathcal{W} , requiring exponential higher frequencies for non integrable many-body quantum systems and thus practically preventing its physical realization.

Noise - In presence of noise Eq. (4) has to be modified: in the following we consider a common scenario however this analysis can be adapted to the specific noise considered. For gaussian white noise, according to Shannon-Hartley theorem the channel capacity is $k_s = \log(1+S/N)$, where S/N is the signal to noise power ratio [19]. Thus, following the same steps as before we obtain that

$$\varepsilon \geq (1 + S/N)^{-\frac{n_s}{D_{\mathcal{W}}}}, \quad (13)$$

and similarly

$$T \geq \frac{D_{\mathcal{W}}}{\Delta\Omega} \frac{\log(1/\varepsilon)}{\log(1 + S/N)}. \quad (14)$$

For small noise to signal ratio ($N/S \ll 1$) the previous bound results in $\varepsilon \gtrsim (N/S)^{n_s/D_{\mathcal{W}}}$ which together with the fact that n_s has to be a polynomial function of $D_{\mathcal{W}}$ show that the control problem is in general exponential sensitive to the problem dimension. However, if one saturates the lower bound on the complexity of the optimal field, i.e. $n_s = D_{\mathcal{W}}$, the sensitivity to Gaussian white noise become linear in the noise to signal ratio. That is, the effects of the noise on the optimal transformation are negligible if the noise level is below the error, $N/S \lesssim \varepsilon$. As requiring the optimal transformation to be more precise than the error on the control signal is somehow unnatural, this relation demonstrate that OC transformations are in general robust with respect to noise, as recently observed [20]. At the same time, for $\varepsilon \lesssim N/S$ this results agrees with the scaling for exact optimal transformations recently found in [9].

Control of unitaries - The aforementioned statements also hold for the generation of unitaries as the differential equation governing the evolution of the time evolution operator $i\hbar\dot{U}(t) = H(t)U(t)$ is formally equivalent to Eq. (1) replacing the density matrix with the time evolution operator $U(t)$, the reference state with the identity operator, and the goal state with the unitary to be generated.

Observability - As any controllable system is also observable by a coherent controller [21], the previous definitions and results can be straightforward applied to the complexity of observing a many-body quantum system with precision ε .

In conclusion, we have shown that if one allows a finite error (both in the goal state and in time) as it typically occurs in any practical application of OC, what can be efficiently simulated can also be optimally controlled and that the optimal solution is in general robust with respect to perturbation on the control field. Notice that the presented results are valid both for open and closed loop OC.

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